A Pareto-Improving Hybrid Policy with Multiclass Network Equilibria

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ABSTRACT
This paper extends the recent work of Song et al. (2014) on Pareto-improving hybrid rationing and pricing policy in general road networks by considering heterogeneous users with different value of time (VOT). Mathematical programming models are proposed for finding a multiclass Pareto-improving pure road space rationing schemes (MPI-PR) and multiclass hybrid rationing and pricing schemes (MHPI and MHPI-S). A numerical example with a 9-nine node multimodal network is provided for comparing both the efficiency and equity of the three proposed policies. We discover that the MHPI-S scheme can bring the most total system delay reduction, the MPI-PR scheme can induce less inequitable than the two hybrid policies, and the MHPI policy is a progressive policy which is appealing to policy makers. The class-specific link flow patterns reveal that different classes of users react differently to the same hybrid policies. In general, low VOT users tend to use transit mode in rationing days, while high VOT users prefer to use car mode in restricted days. In addition, the numerical results also show that multiclass Pareto-improving hybrid schemes yield less delay reduction when compared to its single-class counterparts.

Keywords: Pareto-improving, hybrid policy, rationing, congestion pricing, heterogeneous uses
1. INTRODUCTION

Congestion pricing has been advocated as an effective demand management strategy to reduce traffic congestion and improve system performance, including environment effects, since it was first proposed nearly a century ago (1). Although it has been gaining more support among politicians, transportation officials, and those in legislatures and the media, getting the public to accept congestion pricing is still a major obstacle (2). Hau point out that the marginal cost pricing scheme, which is ubiquitous in the transportation literature, is “most likely doomed to be political failure”, because it makes users worse off compared to the situations without pricing (3).

To make congestion pricing more appealing to the public, the concept of Pareto improvement was introduced to congestion pricing schemes in recent years. Pareto improving congestion pricing schemes increase the social welfare without making any user worse off when compared to the situation without any pricing intervention. Usually, Pareto improving congestion pricing schemes can be achieved through a revenue refunding scheme (4-9).

Besides toll revenue redistribution, Lawphongpanich and Yin proposed a Pareto-improving congestion pricing scheme that leads a general transportation network to Pareto improvement over the status quo (10). The existence of a nonnegative Pareto-improving toll scheme relies on a fact that the original Wardropian user equilibrium (UE) flow distribution may not be strongly Pareto optimal, which means the UE flow distribution may be dominated by another distribution. It is possible to design a charging scheme that evolves flow distributions to the one that is dominating in order to achieve a Pareto improvement. Song et al. (11) proposed anonymous nonnegative Pareto-improving charging schemes with multiple user classes. Wu et al (12) extended the Pareto-improving congestion pricing model to multimodal transportation networks.

Computational experiments in Lawphongpanich and Yin (10) and Song et al. (11) suggest that Pareto-improving tolls are relatively prevalent; however, they may not lead to significant level of improvement. In order to improve the effectiveness of Pareto-improving congestion pricing schemes, researchers turn to combining the congestion pricing policy with other demand management instruments, especially the road space rationing strategy. Road space rationing has been used to mitigate congestion and reduce air pollution in many cities worldwide, such as Athens, Mexico City, Sao Paulo, Beijing and Guangzhou (13). The first hybrid rationing and pricing scheme was proposed by Daganzo (14) to control traffic flow through a bottleneck. He showed that the hybrid scheme can benefit everyone even without revenue redistribution. Later, Daganzo and Garcia (15) extended the hybrid policy to cope with time-dependent bottleneck congestion. They also showed that certain hybrid policy has the potential to achieve Pareto improvement even if not returning the revenue to travelers. Liu et al. (16) presented a simple spatial equilibrium model for a linear monocentric city to investigate the effects of rationing and pricing on morning commuters’ travel cost and modal choice behavior in each location. Song et al. (17) proposed a hybrid policy that integrates congestion pricing and road space rationing, which extends the hybrid strategy proposed by Daganzo (14) from bottleneck level to general congested transportation networks.

In the study of Song et al. (17), all road users are assumed homogeneous and have a single (average) value of time (VOT). However, the value of time is not constant across travelers: it varies by trip purpose (e.g., work versus non-work trips), by sociodemographics, and so on, even for the same traveler at different times and locations (18). This is known as user heterogeneity. Heterogeneity has profound implications for the procedures used to predict users’
path, mode, and departure time choices (19). Because the VOT establishes a connection between
time and money, it plays a critical role in enabling the assignment models and more generally the
demand analysis tools, to provide policy makers with useful information for assessing the
economic and welfare impacts of proposed pricing-based schemes. Ignoring heterogeneity by
using a constant VOT is fundamentally incorrect and would lead to highly biased results (20).

In order to account for the effect of user heterogeneity on hybrid rationing and pricing
policies, this paper extended the work of Song et al. (2014) to the case with a discrete set of user
classes. The primary objective of this paper is to establish mathematical frameworks of designing
an optimal multiclass pure road space rationing policy and optimal multiclass hybrid policies for
a general network and compare both the efficiency and equity of these multiclass policies, as
well as compare the multiclass Pareto improving policies with their single class counterparts. In
particular, optimizations models are presented that determine the rationing ratio, anonymous
Pareto-improving toll vector, and the class-specific link flows.

The remainder of this paper is structured as follows. Section 2 formulates a mathematical
program to design an optimal multiclass Pareto-improving pure road space rationing scheme.
Section 3 formulates a mathematical program to design an optimal multiclass Pareto-improving
hybrid policy. Section 4 compares the efficiency and equity of different multiclass Pareto
improving policies and compares the multiclass hybrid policies with their single class
counterparts on a 9-node multimodal transportation network. The last section concludes the
paper and discusses extensions to this research.

2. A MULTICLASS PARETO-IMPROVING PURE ROAD SPACE RATIONING
   POLICY

In this section, a multiclass Pareto-improving pure rationing is mathematically defined and the
problem of finding such a scheme as a mathematically programs with complementarity
constraints (MPCC) is formulated. To highlight key ideas, it is assumed that the travel demand
for every origin-destination (O-D) pair is fixed. However, travelers have the flexibility to choose
different transportation modes.

2.1 Preliminaries

Similar to many previous studies (21-23), this study represents user heterogeneity by a discrete
set of VOTs, and the users are classified accordingly into multiple classes. For each user class $m$,
let $d^{w,m}$ and $\beta^{w,m}$ denote the travel demand between O-D pair $w$ and the corresponding VOT,
respectively. The problem setting of the multiclass Pareto-improving pure road space rationing is
the same with the single-class Pareto-improving pure road space rationing policy, proposed by
Song et al. (17). Below is the list of sets, subscripts, parameters and variables used in this paper.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>Set of nodes</td>
</tr>
<tr>
<td>$L$</td>
<td>Set of directed links</td>
</tr>
<tr>
<td>$W$</td>
<td>Set of OD pairs</td>
</tr>
<tr>
<td>$K$</td>
<td>Set of user groups</td>
</tr>
<tr>
<td>$M$</td>
<td>Set of user classes</td>
</tr>
<tr>
<td>$A$</td>
<td>Node-arc incidence matrix of the network</td>
</tr>
<tr>
<td>$\bar{A}$</td>
<td>Node-arc incidence matrix of the transit network</td>
</tr>
</tbody>
</table>
\[ L_n \] Set of links in the road network
\[ L_t \] Set of links in the transit network
\[ L_g \] Set of regular links in the road network
\[ L_r \] Set of restricted links in the road network
\[ v_{ij} \] Aggregate traffic flow on link \((i, j)\)
\[ v_{ij}^m \] Class-specific traffic flow on link \((i, j)\)
\[ t_{ij}(v_{ij}) \] Travel time function for link \((i, j)\)
\[ d_{w,m}^{ij} \] Number of passengers of user class \(m\) between OD pair \(w \in W\)
\[ x_{ij}^{w,k,m} \] Link flow of user group \(k\), user class \(m\) on link \((i, j)\)
\[ \alpha \] Rationing ratio
\[ l_{w} \] Direct transit link connecting the origin and destination of OD pairs \(w \in W\)
\[ A_w \] Set of road and transit links for users between OD pair \(w \in W\)
\[ E_w \] An input-output vector
\[ \rho_w \] A vector of Lagrange multipliers
\[ \Phi_{r,w}^{w,1,m} \] Set of links for regular users on path \(r\)
\[ d(w) \] Destination node of OD pair \(w\)
\[ o(w) \] Origin node of OD pair \(w\)
\[ C_{w,m}^{UE} \] Equilibrium travel cost for user class \(m\) before the policy implementation
\[ C_{w,k,m}^{UE} \] Equilibrium travel cost for user group \(k\), class \(m\) between OD pair \(w\)
\[ \tau_{ij} \] Toll imposed on link \((i, j)\)
\[ s_{ij} \] Transit subsidy for transit users on link \((i, j)\)
\[ t_{ij}^0 \] Free-flow travel time (in minute) traversing link \((i, j)\)
\[ b_{ij} \] Capacity (in 100-vehicle) of link \((i, j)\)
\[ VOT \] value of time
\[ UE \] User Equilibrium
\[ OD \] Origin-Destination
\[ KKT \] Karush-Kuhn-Tucker
\[ MPCC \] Mathematical program with complementarity constraints
\[ MFCQ \] Magasarian-Fromovitz constraint qualification

1 In the presence of multiple user classes with different VOT, the system travel disutility can be measured either in time unit (time-based disutility or total system travel time) or in cost or monetary unit (cost-based disutility or total system cost). Absolutely, both time-based and cost-based system disutilities can be regarded as weighted sums of the travel times of all users in the network. The former has a uniform weighting factor equal to unity, while the latter has non-uniform weighting factors equal to the VOT of respective user classes (7). From an economic viewpoint, cost-based disutility is a more appropriate system disutility measure when users have
different VOT. However, in transportation context, time-based disutility has long been accepted as a standard index of system performance. Therefore, in this study, we adopt time-based disutility in all objective functions.

In order to avoid computationally troublesome path enumeration, all the models in this paper are formulated in link-based forms, which can be effectively solved using commercial optimization software. Moreover, in this study, we assumed all link travel costs are positive and separable to avoid a cycle flow in equilibrium solutions, as Newell (25) and Patriksson (26) have proved that a cycle flow cannot be present in an equilibrium solution when travel costs are positive and separable.

2.2. Multiclass User Equilibrium under a pure road space rationing Policy

Based on the model P1 proposed by Song et al (17), the time-based multiclass UE problem under a pure road space rationing policy (MUE-PR) can be formulated as:

\[
\text{MUE-PR:} \quad \min_{(x,v)} \sum_{(i,j) \in L_n} t_{ij} v_i \omega \, d \omega + \sum_{(i,j) \in L_n} t_{ij} v_{ij}
\]

s.t.
\[
A^w x_{ij}^{w,1,m} = (1 - \alpha) E^w d_{ij}^{w,m} \quad \forall w, m
\]
\[
A^w x_{ij}^{w,2,m} = \alpha E^w d_{ij}^{w,m} \quad \forall w, m
\]
\[
v_{ij} = \sum_w \sum_i \sum_m x_{ij}^{w,k,m} \quad \forall (i, j) \in L, w, k, m
\]
\[
x_{ij}^{w,k,m} \geq 0 \quad \forall (i, j) \in L, w, k, m
\]

where constraints (1) and (2) describe flow balance constraints for each class of regular and restricted users, respectively. \( E^w \) is an input-output vector, that is, a vector, with exactly two non-zero components, that specifies the origin and destination of the OD pair \( w \). The component of \( \bar{E}^w \) corresponding to the origin has a value of 1, and the one corresponding to the destination has a value of -1. Constraint (2) states that each class of restricted users can only access the transit network, \( \bar{A} \), which implies that \( x_{ij}^{w,2,m} = 0 \) for all \( (i, j) \in L_n \). Constraint (3) defines the aggregate link flow \( v_{ij} \). Constraints (1)-(4) describe the feasible flow region of a multiclass multimodal network. Observe that objective function of MUE-PR is strictly convex in aggregate link flow \( v_{ij} \) for monotonically increasing link travel time function \( t_{ij} (v_{ij}) \), but linear in class-specific link flow \( v_{ij}^m \) and \( x_{ij}^{w,k,m} \). Thus, under a given pure rationing policy, the equilibrium link flow by user class \( v_{ij}^m \) and \( x_{ij}^{w,k,m} \) are generally not unique, while the aggregate UE link flow \( v_{ij} \) is unique.

In order to demonstrate that Model MUE-PR is equivalent to the UE under a pure road space rationing policy, the Karush-Kuhn-Tucker (KKT) of the program, which is both necessary and sufficient, can be stated as follows,

\[
\beta^m t_{ij} (v_{ij}) + \rho_i^{w,1,m} - \rho_j^{w,1,m} \geq 0 \quad \forall (i, j) \in L_n, w, m
\]
\[
x_{ij}^{w,1,m} \left[ \beta^m t_{ij} (v_{ij}) + \rho_i^{w,1,m} - \rho_j^{w,1,m} \right] = 0 \quad \forall (i, j) \in L_n, w, m
\]
\[
\beta^m t_{ij} + \rho_i^{w,1,m} - \rho_j^{w,1,m} \geq 0 \quad (i, j) = l^w, \forall w, m
\]
\[
x_{ij}^{w,1,m} \left[ \beta^m t_{ij} + \rho_i^{w,1,m} - \rho_j^{w,1,m} \right] = 0 \quad (i, j) = l^w, \forall w, m
\]
\[ \beta^m t_{ij} + \rho^w_{i,j} - \rho^w_{j,i} \geq 0 \quad (i, j) = (v^w, w, m) \tag{9} \]
\[ x_{ij}^{w,m} \left[ \beta^m t_{ij} + \rho^w_{i,j} - \rho^w_{j,i} \right] = 0 \quad (i, j) = (v^w, w, m) \tag{10} \]

and (1)-(4), where \( \rho^w \) is a vector of Lagrange multipliers associated with the flow balance constraints (1) and (2), which are also known as node potentials for OD pair \( w \) (24).

For regular user (user group \( k=1 \)), there are two pairs of complementary constraints, that is, (5)- (8), involved because they are allowed to use all links in set \( A^w \). For class \( m \) users, when a link \( (i, j) \) is utilized, that is, \( x_{ij}^{w,1,m} > 0 \), constraint (6) forces the equation

\[ \beta^m t_{ij} (v_{ij}) + \rho^w_{i,j} - \rho^w_{j,i} = 0 \]

Similarly, if \( x_{ij}^{w,1,m} > 0 \) for regular class \( m \) users on the transit link, then constraint (8) ensures that \( \beta^m t_{ij} + \rho^w_{i,j} - \rho^w_{j,i} = 0 \). Summing up these two sets of equations for each link along a utilized path together yields

\[ \sum_{(i, j) \in \Phi^{w,1,m}} \beta^m t_{ij} (\bullet) = \sum_{(i, j) \in \Phi^{w,1,m}} \rho^w_{i,j} - \rho^w_{j,i} = \rho^w_{d(w)} - \rho^w_{O(w)} \]

where, for OD pair \( w \) and class \( m \) users, \( \Phi^{w,1,m} \) is a set containing links that are available to regular users on path \( r \) and \( d(w) \) and \( o(w) \) denote the destination and origin nodes of OD pair \( w \). \( t_{ij} (\bullet) \) represents both road link and transit link travel times. Thus, constraints (6) and (8) imply that the generalized travel cost of every utilized path equals \( \rho^w_{d(w)} - \rho^w_{O(w)} \) for regular class \( m \) users between OD pair \( w \). When a link \( (i, j) \) is not utilized by class \( m \) users, that is, \( x_{ij}^{w,1,m} = 0 \), constraints (5) and (6) imply that the inequality \( \beta^m t_{ij} (v_{ij}) + \rho^w_{i,j} - \rho^w_{j,i} \geq 0 \) holds. Similarly, if regular class \( m \) users are not using the transit link, that is, \( x_{ij}^{w,1,m} = 0 \), then constraints (7) and (8) ensure that \( \beta^m t_{ij} + \rho^w_{i,j} - \rho^w_{j,i} \geq 0 \). Adding these two inequalities up along a non-utilized path yields \( \sum_{(i, j) \in \Phi^{w,1,m}} \beta^m t_{ij} (\bullet) \geq \rho^w_{d(w)} - \rho^w_{O(w)} \). Therefore, we have the UE conditions for each class of regular users.

For restricted users (user group \( k=2 \)), because they are only allowed to access transit links, one pair of complementarity constraints, that is, (9) and (10), applies. When a link \( (i, j) \) is utilized by class \( m \) users, that is, \( x_{ij}^{w,1,m} > 0 \), constraint (10) forces that \( \beta^m t_{ij} + \rho^w_{i,j} - \rho^w_{j,i} = 0 \). According to the preliminary, there is only one direct link connecting the origin and destination of the OD pair \( w \), and no transit path consists of multiple transit links. There, for restricted class \( m \) users between OD pairs \( w \), \( \beta^m t_{ij} = \rho^w_{d(w)} - \rho^w_{O(w)} \). When a link \( (i, j) \) is not utilized, that is \( x_{ij}^{w,1,m} = 0 \), constraints (9) and (10) imply that the inequality \( \beta^m t_{ij} + \rho^w_{i,j} - \rho^w_{j,i} \geq 0 \) holds and that \( \beta^m t_{ij} \geq \rho^w_{d(w)} - \rho^w_{O(w)} \). Let \( C_{w,k,m}^{UE} = \rho^w_{d(w)} - \rho^w_{O(w)} \), then \( C_{w,k,m}^{UE} \) is the equilibrium travel cost for users of group \( k \) and class \( m \) between OD pairs \( w \). Thus, the UE conditions for each class of restricted users are established.

It is noted that, in the bi-mode transportation network, the equilibrium conditions hold for both road and transit users. Specifically, if regular class \( m \) users use both road and transit modes for a given OD pair, \( w \), then in the multiclass UE, travel cost for that OD pair will be the same for both modes. Based on the above analysis, we prove that the KKT conditions of the
optimization problem MUE-PR are equivalent to the multiclass UE conditions under pure road space rationing schemes.

2.3. Multiclass Pareto-improving pure road space rationing problem
On top of model P2 of Song et al. (17) and model MUE-PR, the multiclass Pareto-improving pure road space rationing problem (MPI-PR) can be formulated as a mathematical program with complementary constraints (MPCC).

\[
\text{MPI-PR: } \min_{\{x, v, \rho, \alpha\}} \sum_{(i,j) \in L} t_{ij} v_{ij} \\
\text{s.t. } (1)-(4) \text{ and } (5)-(10) \\
(1 - \alpha)\left(\rho_{d(w)}^{w,1,m} - \rho_{o(w)}^{w,1,m}\right) + \alpha\left(\rho_{d(w)}^{w,2,m} - \rho_{o(w)}^{w,2,m}\right) \leq C_{w,m}^{UE} \quad \forall w, m
\]

where \(C_{w,m}^{UE}\) is the equilibrium travel cost of class \(m\) users between OD pair \(w\) before the policy implementation, that is, status quo. The objective function minimizes the total system travel cost. Constraints (1)-(10) are the multiclass UE conditions under a pure road space rationing policy. Each class of users may encounter different travel times depending on whether they are restricted on a particular day. Constraint (11) guarantees that when averaged across rationing and regular days, no users are made worse off compared with the status quo. Clearly, the MPI-PR problem always has a feasible solution that is do-nothing solution, which consists of zero rationing ratio value and the multiclass UE under the status quo. If the optimal objective function value is strictly less than the total system travel cost under the status quo, we can conclude that a multiclass Pareto-improving pure road space rationing (MPI-PR) scheme exists.

As formulated, the problem of MPI-PR is an MPCC, a class of problems difficult to solve for mainly two reasons. One is because MPCC violates the Magasarian-Fromovitz constraint qualification (MFCQ), and the other is due to the fact that the feasible region if non-convex (27, 28). An optimization with non-convex feasible region generally contains many local optimal solutions and typically requires a time consuming branch-and-bound scheme to search for a globally optimal solution. When an optimization problem violates MFCQ, one of the weaker constraint qualifications, the KKT conditions may not hold and consequently, cannot be used to verify whether a solution is optimal to the problem (10).

3. MULTICLASS PARETO IMPROVING HYBRID POLICIES
This section describes the mathematical formulation of the proposed Multi-class Pareto-improving Rationing and Pricing Hybrid Policy Problem (MHP),. The formulation presented here also extends the model of P3, P4, and P5 in the paper of Song et al (17) by a discrete set of user classes instead of one single class of users. Note that toll differentiation across user classes is unrealistic and difficult to implement in reality, because users differ from one another in VOT only, which is observationally indistinguishable (21). Therefore, in this section, we only consider congestion pricing schemes with anonymous link flows, which means the same amount of toll is levied on each link for all user classes. In addition, we assume all the link-based tolls and subsidies are nonnegative.

3.1. Multiclass User Equilibrium under Hybrid Rationing and Pricing Policy
The time-based multiclass UE flow distribution in the presence of a hybrid rationing and pricing policy (MUE-HRP) can be formulated by the following mathematical program.
MUE-HRP: \[
\begin{align*}
\min \sum_{(i,j) \in L} \int_0^{v_j} t_j(\omega) \, d\omega + \sum_{(i,j) \in L} t_{ij}v_j + \sum_{(i,j) \in L} \sum_{m \in M} \tau_{ij}x_{ij}^{w,2,m} \\
\text{s.t. } A^w x_{w,1,m} = (1-\alpha)E^w d_{w,m} \quad \forall w \\
A^w x_{w,2,m} = \alpha E^w d_{w,m} \quad \forall w \\
v_j = \sum_i \sum_k \sum_m x_{ij}^{w,k,m} \\
x_{ij}^{w,k,m} \geq 0 \quad \forall (i,j) \in L, w, k, m
\end{align*}
\] (12)

where constraints (12) and (13) describe flow balance constraints for regular and restricted users, respectively. For restricted users, they have to pay nonnegative anonymous toll $\tau_{ij}$ if they choose to use the restricted links $(i,j) \in L_r$.

Constraints (12)-(15) describe the feasible flow region of a multiclass multimodal network, which is convex. Also note that the objective function of MUE-HRP is strictly convex in aggregate link flow $v_j$ for monotonically increasing link travel time function $t_j(v_j)$, but linear in class-specific link flow $v_j^m$ and $x_{ij}^{w,k,m}$. Thus, under a hybrid rationing and pricing policy, the equilibrium link flow by user class $v_j^m$ and $x_{ij}^{w,k,m}$ are generally not unique, while the aggregate UE link flow $v_j$ is unique. In order to illustrate that the MUE-HPR problem is equivalent to the multiclass UE under a hybrid rationing and pricing policy, the KKT conditions of the MUE-HRP can be expressed as:

\[
\begin{align*}
\beta^nt_j\left(v_j\right) + \rho_j^{w,k,m} - \rho_j^{w,k,m} & \geq 0 \quad \forall (i,j) \in L, w, k, m \\
x_{ij}^{w,k,m} \left[ \beta^nt_j\left(v_j\right) + \rho_j^{w,k,m} - \rho_j^{w,k,m} \right] & \geq 0 \quad \forall (i,j) \in L, w, k, m \quad (16) \\
\beta^nt_j\left(v_j\right) + \rho_j^{w,k,m} - \rho_j^{w,k,m} & \geq 0 \quad (i,j) = L_r, \forall w, k, m \quad (17) \\
x_{ij}^{w,k,m} \left[ \beta^nt_j\left(v_j\right) + \rho_j^{w,k,m} - \rho_j^{w,k,m} \right] & = 0 \quad (i,j) = L_r, \forall w, k, m \quad (18) \\
\beta^nt_j\left(v_j\right) + \rho_j^{w,1,m} - \rho_j^{w,1,m} & \geq 0 \quad \forall (i,j) \in L, w, m \quad (19) \\
x_{ij}^{w,k,m} \left[ \beta^nt_j\left(v_j\right) + \rho_j^{w,1,m} - \rho_j^{w,1,m} \right] & \geq 0 \quad \forall (i,j) \in L, \forall w, m \quad (20) \\
\beta^nt_j\left(v_j\right) + \rho_j^{w,2,m} - \rho_j^{w,2,m} & \geq 0 \quad \forall (i,j) \in L, w, m \quad (21) \\
x_{ij}^{w,k,m} \left[ \beta^nt_j\left(v_j\right) + \rho_j^{w,2,m} - \rho_j^{w,2,m} \right] & \geq 0 \quad \forall (i,j) \in L, \forall w, m \quad (22) \\
\end{align*}
\] (22)

and (12)-(15), where $\rho^w$ is a vector of Lagrange multipliers (node potentials) associated with the flow balance constraints (12) and (13). Definitely, the KKT conditions of MUE-HRP contain four pairs of complementary constraints (16)-(23). Carrying out the same procedure in the MUE-PR problem, it can be easily proved that the above KKT conditions are equivalent to the multiclass UE conditions under a hybrid rationing and pricing policy. For each class of regular users (user group $k=1$), as they are free to access all links in set $A^w$, UE conditions can be established using complementarity constraints (16)-(19). While for each class of restricted users (user group $k=2$), as they have to pay congestion tolls to access the restricted links; tolled UE conditions can be obtained from complementary constrains (16)-(19), (22) and (23).

### 3.2. Multiclass Pareto-Improving Hybrid Rationing and Pricing Policy
Based on the formulation of MPI-PR and MUE-HRP, the multiclass Pareto-Improving hybrid rationing and pricing problem (MHPI) can be formulated without difficulty. In the MHPI problem, the restriction ratio $\alpha$, the set of restricted links, and their corresponding nonnegative anonymous toll rates $\tau$ for restricted users have to be specified to minimize the time-based total system delay. Following the same logic of Model P4 in Song et al. (17), we formulate the MHPI problem as an MPCC in the following:

$$\min_{(v,\rho,\alpha,\tau)} \sum_{(i,j) \in L_n} t_{ij}(v_{ij})v_{ij} + \sum_{(i,j) \in L_r} t_{ij}v_{ij}$$

s.t.
$$\beta^n t_{ij}(v_{ij}) + \rho_i^{w,1,m} - \rho_j^{w,1,m} \geq 0 \quad \forall (i,j) \in L_n, w, m \quad (24)$$
$$\beta^n t_{ij}(v_{ij}) + \rho_i^{w,1,m} - \rho_j^{w,1,m} \geq 0 \quad \forall (i,j) \in L_n, w, m \quad (25)$$
$$\beta^n t_{ij}(v_{ij}) + \rho_i^{w,2,m} - \rho_j^{w,2,m} \geq 0 \quad \forall (i,j) \in L_n, w, m \quad (26)$$
$$\beta^n t_{ij}(v_{ij}) + \rho_i^{w,2,m} - \rho_j^{w,2,m} = 0 \quad \forall (i,j) \in L_n, w, m \quad (27)$$
$$\beta^n t_{ij} + \rho_i^{w,k,m} - \rho_j^{w,k,m} \geq 0 \quad (i,j) = l^w_i, \forall w, k, m \quad (28)$$
$$\beta^n t_{ij} + \rho_i^{w,k,m} - \rho_j^{w,k,m} = 0 \quad (i,j) = l^w_i, \forall w, k, m \quad (29)$$
$$\rho_i^{w,1,m} - \rho_j^{w,1,m} \leq \beta^m C^u_{w} \quad \forall w, m \quad (30)$$
$$\rho_i^{w,2,m} - \rho_j^{w,2,m} \leq \beta^m C^u_{w} \quad \forall w, m \quad (31)$$

and (12)-(15).

Constraints (24)-(29) are 3 pair of complementary conditions, which treat all road links as potential restricted links. If the toll rate for restricted users is strictly positive on link $(i,j)$, the link belongs to $L_r$; otherwise, $(i,j) \in L_k$. Constraint (30) is the Pareto-improving conditions, which means when averaged across restricted days and free days; no user is made worse off compared to the status quo. When the optimal objective value of MHPI is strictly less than that under the status quo, the solution vector is a multiclass Pareto-improving hybrid scheme. Although, the existence of a strictly Pareto-improving solution cannot be guaranteed, we can conclude that the MHPI problem has at least one feasible solution. Because if setting a sufficiently large toll on all road links for restricted users so that they cannot afford using the road network, the MHPI problem degraded to MPI-PR problem, and we have proved that the MPI-PR problem always has a feasible solution in the last section.

### 3.3 The Multiclass Hybrid Pareto-Improving Policy with Transit Subsidy

The above multiclass hybrid Pareto improvement is attained without a toll revenue redistribution procedure. In the literature, toll revenue redistribution has been proved an effective way to achieve Pareto improvement in theory. This enlightens us to incorporate a certain revenue redistribution plan in the MHPI scheme, so as to examine whether more substantial improvement can be made. Similar to Song et al. (17), we adopt a transit subsidy plan that directly uses the toll revenue to subsidize each class of transit users by reducing their transit fares. We formulated the multiclass hybrid policy with transit subsidy (MHPI-S) problem as the following MPCC,
MHPI-S: \[
\min_{\{x,v,\rho,a,s\}} \sum_{(i,j) \in L_a} t_{ij} (v_{ij}) v_{ij} + \sum_{(i,j) \in L_a} t_{ij} v_{ij}
\]
s.t. (12)-(15), (24)-(27), (30) and (31)
\[
\beta^m \left( t_{ij} - s_{ij} \right) + \rho_{ij}^{w,k,m} - \rho_{ij}^{w,k,m} \geq 0, \quad (i,j) = I^w, \quad \forall w, k, m \tag{32}
\]
\[
\sum_{(i,j) \in L_a} s_{ij} v_{ij} \leq \sum_{(i,j) \in L_a} \sum_{(i,k) \in L_a} \sum_{m \in M} t_{ij} x_{ij}^{w,2,m} \tag{34}
\]
\[
s_{ij} \geq 0, \quad \forall (i,j) \in L_i \tag{35}
\]
where \(s_{ij}\) denotes an anonymous nonnegative subsidy, that is, a reduction of transit fare, to each class transit users using link \((i,j)\). Constraint (31) and (35) require both tolls and subsidies to be nonnegative. Nonnegative anonymous tolls, \(\tau_{ij}\), are levied on each class of restricted users who choose to use restricted links, whereas subsidies, \(s_{ij}\), are provided to both regular and restricted users using the transit links. To guarantee the whole system can be self-sustained, we use constraint (34) to force the total subsidy to transit users must less than or equal to the total toll revenue collected.

4. NUMERICAL EXAMPLES
To explore the existence and properties of MPI-PR and MHPI, we use the same example as Song et al. (17), as shown in Figure 1. To be self-contained, here we restate the characteristics of this example. The 9-node multi-mode network contains four OD pairs [\((1,3), (1,4), (2,3) and (2,4)\)], 18 auto-links and 4 transit links. The aggregate demand of each OD pair (in 100 passengers/hour) is also shown in Figure 1. The link performance functions associated with auto-links are assumed to follow Bureau of Public Roads (BPR) function,
\[
t_{ij} (v_{ij}) = t_{ij}^0 \left[ 1 + 0.15 \left( \frac{v_{ij}}{b_{ij}} \right)^4 \right], \quad \text{where} \quad t_{ij}^0 \quad \text{is the free-flow travel time (in minutes) and} \quad b_{ij} \quad \text{is the capacity (in 100 vehicles) of link} \quad (i,j). \quad \text{The parenthesis near auto-link} \quad (i,j) \quad \text{in the figure}
\]
denotes \(\left( t_{ij}^0, b_{ij} \right)\). Four transit links (dotted lines) directly connect the origin and destination nodes of each OD pair. Because the travel cost on transit link \((i,j)\) is represented by its generalized travel cost, it is reasonable to assume \(t_{ij}\) to be higher than the equilibrium travel cost (time) on the road network for the same OD pair. The original network-related settings were kept, two classes of users, m1 and m2, were introduced with VOTs of 0.6 $/minute and 1.4 $/minute, respectively. A uniform 50% and 50% split was assumed between the low- and high-VOT users for each O-D pair.

For MPI-PR, MHPI and MHPI-S are MPCC, we revise the manifold suboptimization algorithm proposed by Lawphongpanich and Yin (10) to find strongly stationary solutions. The algorithm is implemented on GAMS (29). Nonlinear subproblems involved are solved by the solver of CONOPT (30). Due to the limitation of software license, some computations are conducted on the NEOS Server (31). According to the above analysis, the optimal solution of MPI-PR is a feasible solution of MHPI. Therefore, in order to solve MHPI more efficiently, we use the resulting flow distribution of MPI-PR as an initial solution of MHPI problem. Having
solved all the models proposed in this paper, it is found that all the problems tested are solved very efficiently. In fact, the longest elapsed time reported by GAMS is within 1 second.

![Network Layout](image_url)

**Figure 1** The network layout for demonstrating example.

We assume that the travel cost of any given transit link between OD pair \( w \in W \) is exactly 150% of its corresponding equilibrium travel time using only road network. Having solved problem MUE, MPI-PR, MHPI and MHPI-S, we summarized the results in TABLE 1-5.

The first scenario compares flow distributions and the system performance of the multiclass user equilibrium (MUE) and three multiclass Pareto improving policies. TABLE 1 and TABLE 2 provide the link flow distribution under different policies. More specifically, the first four rows of each table show the number of users using the direct transit links, which also coincide with the number of transit users for the corresponding OD pairs. Note that, in the last column of TABLE 2, the first four rows represent subsidies to transit users, which are shown as negative values in the “Toll” column of MHPI-S scheme. As we have proved, the aggregate link flow of MUE is unique and the class-specific link flow of MUE is not unique. In TABLE 1, we only display the aggregate link flow of MUE. However, in order to gain more insights from the multiclass policies; we list both the unique aggregate link flow distribution and one of the class-specific link flow distributions under different policies.

TABLE 3 provides (average) class-specific equilibrium travel costs, the optimal rationing/restriction ratio (\( \alpha \)), total system delay and delay reduction under different policy schemes. The “delay reduction (% of max)” refers to the ratio between the reduction in travel delay and the maximum amount possible, that is, the difference of system delay under MUE and multiclass system optimum (MSO).

When transit travel costs are 150% of the corresponding equilibrium travel costs using the road network, the total system delay is 490 409.6 passenger minutes and 588 380.0 passenger...
minutes under MSO and MUE conditions, respectively. As shown in TABLE 3, all the three policies generate Pareto improvement. The hybrid policies provide substantially better system performance than the pure rationing policy and the MHPI-S policy provides the best system performance.

**TABLE 1 Flow Distribution of MUE and Multiclass Pareto-Improving Pure Rationing**

<table>
<thead>
<tr>
<th>Link</th>
<th>$v^{E}$</th>
<th>$v^{k,1}$</th>
<th>$v^{k,1,m1}$</th>
<th>$v^{k,1,m2}$</th>
<th>$v^{E2}$</th>
<th>$v^{k,2,m1}$</th>
<th>$v^{k,2,m2}$</th>
</tr>
</thead>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>8.988</td>
<td>4.494</td>
<td>4.494</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>8.988</td>
<td>4.494</td>
<td>4.494</td>
</tr>
<tr>
<td>(2, 3)</td>
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<td>11.984</td>
<td>5.992</td>
<td>5.992</td>
</tr>
<tr>
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<td>0.000</td>
<td>0.000</td>
<td>14.980</td>
<td>7.490</td>
<td>7.490</td>
</tr>
<tr>
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<td>22.937</td>
<td>14.508</td>
<td>4.002</td>
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<td>0.000</td>
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<td>27.516</td>
<td>17.010</td>
<td>10.506</td>
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<td>15.732</td>
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<td>0.000</td>
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<td>0.000</td>
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</table>

We can further observe that most of class-specific average equilibrium travel cost under MHPI is lower than that under the pure rationing scheme, except for OD pair (1, 4). All of the class-specific average equilibrium travel cost under MHPI-S is lower than that under the pure rationing scheme.

More importantly, from TABLE 1 and TABLE 2, we also find that class m2 users (high VOT users) who are originally forced to use transit links under the pure rationing scheme, for example, link (1, 3) and link (2, 4), choose to pay tolls to access road networks under hybrid policies, which indicates that certain class of users do benefit from the flexibility brought by the hybrid policy. However, in the same case, all of class m1 users (low VOT users) remain using the transit lines in restricted days under MHPI and MHPI-S schemes, which indicate that different classes of users react differently to the same hybrid policies. Another interesting scenario can be seen under the MHPI-S scheme. A fraction of regular class m1 users use the transit links, for
example, link (1, 3), which indicate that the subsidy scheme is substantially attractive to the low VOT travelers, so that they choose to use transit even in unrestricted days. This scenario shows the potential that transit mode share can be improved by a hybrid policy with transit subsidies.
<table>
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<tr>
<th>Link</th>
<th>$v_{k1}^{1}$</th>
<th>$v_{k1,m1}^{1}$</th>
<th>$v_{k1,m2}^{1}$</th>
<th>$v_{k2}^{2}$</th>
<th>$v_{k2,m1}^{2}$</th>
<th>$v_{k2,m2}^{2}$</th>
<th>Toll ($)</th>
<th>$v_{k1}^{1}$</th>
<th>$v_{k1,m1}^{1}$</th>
<th>$v_{k1,m2}^{1}$</th>
<th>$v_{k2}^{2}$</th>
<th>$v_{k2,m1}^{2}$</th>
<th>$v_{k2,m2}^{2}$</th>
<th>Toll ($)</th>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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</tr>
<tr>
<td>(8,3)</td>
<td>5.766</td>
<td>4.116</td>
<td>1.650</td>
<td>4.690</td>
<td>0.000</td>
<td>4.690</td>
<td>9.088</td>
<td>10.072</td>
<td>0.579</td>
<td>9.493</td>
<td>12.851</td>
<td>0.000</td>
<td>12.851</td>
<td>4.359</td>
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<tr>
<td>(8,4)</td>
<td>39.181</td>
<td>11.688</td>
<td>27.493</td>
<td>6.131</td>
<td>0.000</td>
<td>6.131</td>
<td>9.710</td>
<td>33.188</td>
<td>9.493</td>
<td>23.695</td>
<td>8.590</td>
<td>0.000</td>
<td>8.590</td>
<td>22.165</td>
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<tr>
<td>(8,7)</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.026</td>
<td>0.000</td>
<td>0.026</td>
<td>3.209</td>
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<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
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<td>3.813</td>
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<tr>
<td>(9,7)</td>
<td>29.749</td>
<td>21.089</td>
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<td>1.660</td>
<td>0.000</td>
<td>1.660</td>
<td>13.954</td>
<td>24.472</td>
<td>24.472</td>
<td>0.000</td>
<td>5.122</td>
<td>0.000</td>
<td>5.122</td>
<td>14.042</td>
</tr>
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<td>(9,8)</td>
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<td>0.000</td>
<td>0.000</td>
<td>10.847</td>
<td>0.000</td>
<td>10.847</td>
<td>1.305</td>
<td>0.000</td>
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<td>0.000</td>
<td>21.441</td>
<td>0.000</td>
<td>21.441</td>
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</tr>
</tbody>
</table>
### TABLE 3 Summary of MUE, MPI-PR, MHPI and MHPI-S

<table>
<thead>
<tr>
<th>OD pair</th>
<th>MUE</th>
<th>MPI-PR</th>
<th>MHPI</th>
<th>MHPI-S</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C^{UE}_{w,m_1}$</td>
<td>$C^{UE}_{w,m_2}$</td>
<td>$C^{MP-PR}_{w,m_1}$</td>
<td>$C^{MP-PR}_{w,m_2}$</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>27.378</td>
<td>63.882</td>
<td>25.065</td>
<td>0.916</td>
</tr>
<tr>
<td>(1, 4)</td>
<td>27.197</td>
<td>63.459</td>
<td>24.610</td>
<td>0.905</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>21.134</td>
<td>49.314</td>
<td>20.672</td>
<td>0.978</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>20.953</td>
<td>48.891</td>
<td>20.402</td>
<td>0.974</td>
</tr>
<tr>
<td>Optimal Rationing Ratio (α)</td>
<td>0.0000</td>
<td>0.2996</td>
<td>0.3127</td>
<td>0.3672</td>
</tr>
<tr>
<td>Total system delay ($)</td>
<td>588380.0</td>
<td>534709.6</td>
<td>524567.8</td>
<td>505277.3</td>
</tr>
<tr>
<td>Delay reduction (% of max)</td>
<td>0</td>
<td>45.69</td>
<td>65.13</td>
<td>84.82</td>
</tr>
</tbody>
</table>

### TABLE 4 Mode Split of Multiclass Pure Rationing and Multiclass Hybrid Policies

<table>
<thead>
<tr>
<th>OD pairs</th>
<th>Multiclass pure rationing policy</th>
<th>Multiclass hybrid policy</th>
<th>Multiclass hybrid policy with subsidy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Demand of class m1 (100 passengers/hour)</td>
<td>Demand of class m2 (100 passengers/hour)</td>
<td>Demand of class m1 (100 passengers/hour)</td>
</tr>
<tr>
<td>Optimal Rationing Ratio (α)</td>
<td>0.2996</td>
<td>0.3127</td>
<td>0.3672</td>
</tr>
</tbody>
</table>
### TABLE 5 Summary of SHPI, MHPI, SHPI-S and MHPI-S

<table>
<thead>
<tr>
<th>OD pair</th>
<th>SHPI (single-class)</th>
<th>MHPI (multiclass)</th>
<th>SHPI-S (single-class)</th>
<th>MHPI-S (multiclass)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average Equilibrium cost ($)</td>
<td>Average equilibrium cost ($)</td>
<td>Average Equilibrium cost ($)</td>
<td>Average equilibrium cost ($)</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>43.131</td>
<td>25.065</td>
<td>58.485</td>
<td>35.529</td>
</tr>
<tr>
<td>(1, 4)</td>
<td>42.380</td>
<td>24.610</td>
<td>57.423</td>
<td>35.634</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>33.760</td>
<td>20.672</td>
<td>48.234</td>
<td>30.655</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>34.922</td>
<td>20.402</td>
<td>47.604</td>
<td>33.090</td>
</tr>
<tr>
<td>Optimal Rationing Ratio ($\alpha$)</td>
<td>0.3449</td>
<td>0.3127</td>
<td>0.4430</td>
<td>0.3672</td>
</tr>
<tr>
<td>Total system delay ($)</td>
<td>505 144.5</td>
<td>524 567.8</td>
<td>499 107.3</td>
<td>505 277.3</td>
</tr>
<tr>
<td>Total toll ($)</td>
<td>61 041.1</td>
<td>442 83.6</td>
<td>38 043.7</td>
<td>61 227.3</td>
</tr>
<tr>
<td>Total subsidy ($)</td>
<td>0.000</td>
<td>0.000</td>
<td>35 590.6</td>
<td>31 571.5</td>
</tr>
<tr>
<td>Delay reduction (% of max)</td>
<td>84.96</td>
<td>65.13</td>
<td>91.12</td>
<td>84.82</td>
</tr>
</tbody>
</table>
Then, the equity issues of the proposed multiclass pure rationing policy and the multiclass hybrid policies are explored. From TABLE 3, we can easily obtain the frequencies and cumulative distributions of three ratios, 
\[ C_{w,m}^{\text{MPI-PR}} / C_{w,m}^{\text{UE}}, C_{w,m}^{\text{MHPI}} / C_{w,m}^{\text{UE}}, \text{ and } C_{w,m}^{\text{MHPI-S}} / C_{w,m}^{\text{UE}}, \]
where \( C_{w,m}^{\text{MPI-PR}}, C_{w,m}^{\text{MHPI}}, \text{ and } C_{w,m}^{\text{MHPI-S}} \) are average equilibrium cost under MPI-PR, MHPI and MHPI-S schemes, respectively. Then, we can calculate the Gini coefficients associated with the ratio of \( C_{w,m}^{\text{MPI-PR}} / C_{w,m}^{\text{UE}}, C_{w,m}^{\text{MHPI}} / C_{w,m}^{\text{UE}}, \text{ and } C_{w,m}^{\text{MHPI-S}} / C_{w,m}^{\text{UE}} \) are 0.1003, 0.1624, and 0.1060, respectively, suggesting that MPI-PR scheme induce less inequality than both the hybrid policies, and incorporating the toll revenue redistribution procedure can improve the equity level of MHPI scheme substantially. These scenarios are all accordance with the consensus that rationing policy are more equitable than pricing-based policies and combining revenue refunding can improve the equity of pricing schemes. Moreover, it is worth noting that, under MPI-PR scheme and MHPI-S policy, the ratios \( C_{w,m}^{\text{MPI-PR}} / C_{w,m}^{\text{UE}} \) and \( C_{w,m}^{\text{MHPI-S}} / C_{w,m}^{\text{UE}} \) is the same for each class of users and all OD pairs, while under the MHPI policy, the ratio \( C_{w,m}^{\text{MHPI}} / C_{w,m}^{\text{UE}} \) of low VOT users are less than that of high VOT users for all OD pairs. This scenario can partially explain the larger Gini coefficient value of MHPI scheme. More importantly, it indicates that low-VOT users tend to be better off than the high-VOT users under the MHPI scheme, which makes MHPI scheme a progressive policy and most attractive scheme to policy makers among the proposed three schemes (32).

To further investigate the impact of different policies on the change of mode split, TABLE 4 shows the class-specific mode split under pure rationing and hybrid policies. From TABLE 4, we discover that the multiclass pure rationing policy imposes a uniform diversion rate to transit for all OD pairs, which ignores the difference in network traffic conditions faced by users from different OD pairs, and may inevitably cause inefficient allocation of road capacity resources. Intuitively, the multiclass hybrid policies will induce more car demands than their multiclass pure rationing counterpart. But it is worth noting that the optimal rationing ratios are not the same under the three policies and the mode split change also depends on the optimal rationing ratios under different policies. Therefore, the general tendency for the change of mode split is difficult to depict. More specifically, in this case, on the aggregate level (sum all class of users), even with higher optimal restriction ratios, MHPI policy yields more car demand splits in half OD pairs, e.g. OD pair (1, 3) and (2, 4), and MHPI-S policy yields more car demands splits in all the OD pairs, which indicate hybrid policies are more flexible and effective in managing multimode network mobility. Furthermore, in the class-specific level, different classes of users behave differently under the hybrid policies. Generally speaking, higher proportion of class m2 users (high VOT) choose car mode than class m1 users r (low VOT) under hybrid policies. In particular, under MHPI-S policy, all the class m2 users, unrestricted and restricted, choose car mode, meanwhile, a fraction of unrestricted class m1 users choose the transit mode.

In TABLE 5, we compared flow distributions and the system performance of the multiclass hybrid policies with its single class counterpart. From TABLE 5, we observe that, both MHPI and MHPI-S schemes yields less delay reduction compared to their single-class counterpart. The existence and effectiveness multiclass Pareto-improving hybrid schemes not only depend on the network configurations but also on demographic features (e.g., VOT values) of the population. It is also worth noting that we solve all the models by manifold suboptimization algorithm, which can only guarantee converging to strongly stationary solutions, not necessarily global optimal solutions. Thus, the truly global optimal solutions of all
these models may yield less travel delay.

5. CONCLUSIONS
This study extended hybrid rationing and pricing Pareto-improving policy for homogeneous users to heterogeneous users. The heterogeneous users refer to a discrete set of classes, each one with a different VOT. The numerical results shows that, multiclass Pareto improving pure rationing policies and multiclass hybrid Pareto improving policies are all exist. Like the single-class hybrid policies, the multiclass hybrid policies (MHPI and MHPI-S) also provide greater flexibility than multiclass pure rationing (MPR-PR) policy. Under the MPR-PR policy, all class-specific travel demands are forced to use transit mode at the same ratio regardless of the traffic conditions, which may cause inefficient allocation of road resources. On the contrast, under multiclass hybrid policies, although the restriction ratio is still the same for all OD pairs and all classes of users, link-based nonnegative anonymous tolls are introduced to adjust roadway demands for different OD pairs and different user classes.

Comparing the equity level of the three policies, we discover that pure rationing policy induce less inequalities than two hybrid policies and hybrid policies with transit subsidies can improve the original MHPI policy substantially. MHPI policy is a progressive policy since it favors the disadvantaged travelers (e.g. travelers with low VOT).

Different classes of users react differently to the same hybrid policies. As the numerical example shows, low VOT users tend to use transit mode in rationing days while restricted high VOT users may be better off paying tolls to access the road network on restricted days instead of taking transit. Furthermore, under the hybrid Pareto-improving policy with subsidies, low VOT users may use transit mode even in the unrestricted days. Although tolls are charged on restricted users only, their route choice may influence both the mode choice and route choice of non-restricted users, hence achieve better system performance.

The numerical results also illustrate that, compared to the single-class hybrid polices, multiclass hybrid policies yields less delay reduction. The existence and effectiveness multiclass Pareto-improving hybrid schemes not only depend on the network configurations but also on demographic features (e.g., VOT values) of the population.

In this paper, we assume that travel demand is deterministic to facilitate the presentation of key ideas. The theory of elastic demand can be similarly established. Other extensions to the model include adopting continuous VOT distributions instead of a discrete set of classes, relaxing the strict Pareto-improving conditions to approximate Pareto-improving conditions, considering more realistic network configurations, such as variable transit travel times, intermodal scenarios, and applying the models to large scale real networks.

All in all, the multiclass hybrid Pareto-improving rationing and pricing schemes offer an effective and equitable tool to manage network mobility, traffic congestion and environmental quality. Its various potential applications remain largely unexplored.

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